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THEORETICAL MODELS OF SURFACE HEAT TREATMENT OF PRODUCTS IN SOLAR FURNACES. 2. NONUNIFORM RADIANT-ENERGY FLUX

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The nonlinear and two-dimensional problem of heating of a plane plate in a solar furnace with a Gaussian distribution of the radiant heat flux over the surface has been reduced to the one-dimensional problem by introducing a nondifferentiable parameter dependent on the radial coordinate. Using the modified Goodman heat-balance integral method, we have obtained estimates of the minimum heat flux necessary for fusion of the surface of the plate and the limiting diameter of the melt hole. An example of calculating the dynamics of fusion of the surface of a ceramic plate is presented.

Introduction. In the first part of this work [1], it has been shown that the processes of heat treatment of products in solar furnaces can be calculated by integral methods of calculation of the nonlinear problems of heat conduction in the cases where a radiant-energy flux is uniformly distributed over the focal-spot area. These methods are, as a rule, applied to one-dimensional (plane, cylindrical, and spherical) problems. However, in solar furnaces, such an energy distribution occurs only over a small part of the focal-spot area (unless additional technical measures are taken). Because of the special properties of paraboloid concentrators of solar radiation and errors in their production, the radiant energy flux is nonuniformly distributed over the focal-spot area by a nearly Gaussian law; in such cases, one-dimensional models are inapplicable. We have made an effort to modify one integral method for solution of the nonlinear problem of heating of a plate by a nonuniform radiant-energy flux.

Method of Approximate Solution of the Problem on Heating of a Plate by a Nonuniform Flux. In the presence of axial symmetry, the heat-conduction equation for a material with constant thermophysical properties has the form

$$\frac{\partial T}{\partial t} = a^2 \left(\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} \right), \quad a^2 = \frac{\lambda}{\rho c} . \tag{1}$$

The heat flux incident on the surface of a plate is described by the Gaussian law

$$q_i = q_{i0} \exp \left(- \frac{r^2}{r_*^2} \right). \tag{2}$$

The success in solving the problem by an approximate integral method essentially depends on how exactly the form of the approximating function is predicted. In this connection, we note that when there is no radiation from the surface, the heat flux penetrating into the plate is distributed by the exponential law (2); therefore, the heat flux in the plate in the z direction can be represented in the form of a product:

$$q_z = q_z' (t, r, z)\, \Phi (r) , \quad \text{where} \quad \Phi = \exp \left(- \frac{r^2}{r_*^2} \right). \tag{3}$$

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Expression (3) is exact if it is proposed to find q_z' using the initial heat-conduction equation (1), which, apparently, can be written in terms of the heat flux:

$$\frac{\partial q_z}{\partial t} = a^2 \left(\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial q_z}{\partial r} + \frac{\partial^2 q_z}{\partial z^2} \right), \quad q_z = -\lambda \frac{\partial T}{\partial z} . \tag{4}$$

We will consider (3) as a formula for approximation of the heat flux, in which the function $\Phi(r)$ reflects the main, strongest dependence of the heat flux on the radial coordinate. Unlike the exact expression, the coordinate r in q_z' will be considered as a parameter, i.e., we will not differentiate q_z' with respect to r. This method is similar to the introduction of varied and nonvaried functions in the method implementing the "principle of minimum entropy production" of Prigogine [2]. The temperature approximation is found by integrating (3) with respect to z. If the adopted condition is met, substitution of (3) into (4) gives

$$\frac{\partial q_z'}{\partial t} = a^2 \left[-\frac{\alpha(r)}{r_*^2} q_z' + \frac{\partial^2 q_z'}{\partial z^2} \right], \quad \alpha(r) = -\frac{r_*^2}{r\Phi(r)} \frac{d}{dr} \left(r \frac{d\Phi(r)}{dr} \right). \tag{5}$$

Let us introduce the approximation form parameter $\delta(r, t)$ — "heat-penetration depth," which is characterized by the fact that at $z \geq \delta(r, t)$ the heat flux is equal to zero and the temperature is equal to the initial temperature ($T = 0$) and is determined by the formula

$$T = \frac{\Phi(r)}{\lambda} \int_z^{\delta} q_z' dz . \tag{6}$$

Integrating (5) with respect to z and taking account of (6), we can obtain an equation analogous to (5) for the temperature:

$$\frac{\partial T}{\partial t} = a^2 \left[-\frac{\alpha(r)}{r_*^2} T + \frac{\partial^2 T}{\partial z^2} \right]. \tag{7}$$

Thus, we have obtained the one-dimensional equation of heat conduction with heat sink that reflects the spreading of the heat flux from the point of its supply and is proportional to αT.

The boundary condition on the heated surface ($z = 0$) is

$$q_{i0}\Phi(r) = -\left(\lambda \frac{\partial T}{\partial z} \right)_{z=0} + \varepsilon\sigma T_0^4 . \tag{8}$$

The heat-balance integral method of Goodman [3] and its variants [4] give good results precisely for one-dimensional equations. We take the approximation

$$T = T_0 (1 - \xi)^n , \quad \xi = \frac{z}{\delta}$$

and integrate Eq. (7) with respect to the coordinate z between the limits $0 \leq z \leq \delta(r, t)$:

$$\frac{d}{dt} \left(\frac{T_0 \delta}{n+1} \right) = a^2 \left(-\alpha \frac{T_0 \delta}{n+1} + \frac{n T_0}{\delta} \right). \tag{9}$$

Boundary condition (8) will take the form

$$q_{i0}\Phi(r) = \frac{n\lambda T_0}{\delta} + \varepsilon\sigma T_0^4 . \tag{10}$$

If the index n is given, (9) and (10) solve the problem completely since they involve two unknown functions: T_0 and δ. To decrease the error in the approximate calculations one uses additional relations by introducing new form parameters [5]. We will consider the index n to be a form parameter and, to determine its time dependence, assume that the differential equation (7) is exactly fulfilled at the point $z = 0$:

$$\frac{dT_0}{dt} = a^2 \left[-\frac{\alpha T_0}{r_*^2} + \frac{n\,(n-1)\,T_0}{\delta^2} \right]. \tag{11}$$

We write the equations of the problem (11)–(13) in dimensionless form, introducing the following notation:

$$\tau = \frac{t}{t_*};\quad p = \frac{r}{r_*};\quad \eta = \frac{\delta}{l_*};\quad \theta = \frac{T_0}{T_*};\quad T_* = \left(\frac{q_{i0}}{\varepsilon\sigma}\right)^{1/4};\quad l_* = \frac{\lambda T_*}{q_{i0}};\quad t_* = \frac{l_*^2}{a^2};\quad R = \frac{r_*}{l_*}.$$

We note that in this case, the temperature scale is the so-called "radiation-equilibrium temperature" at which the entire heat incident on the surface is reradiated into space. As a result we obtain

$$\frac{d}{d\tau}\left(\frac{\theta\eta}{n+1}\right) = -\frac{\alpha\,(p)\,\theta}{R^2}\,\frac{\eta}{n+1} + \frac{n\theta}{\eta}, \tag{12}$$

$$\frac{d\theta}{d\tau} = -\frac{\alpha\,(p)\,\theta}{R^2} + \frac{n\,(n-1)\,\theta}{\eta^2}, \tag{13}$$

$$\frac{n\theta}{\eta} = \Phi\,(p) - \theta^4. \tag{14}$$

This system contains two parameters: $\alpha(p)/R^2$ and $\Phi(p)$. By changing the variables

$$\omega = \tau\Phi^{3/2},\quad \varphi = \theta\Phi^{-1/4},\quad \psi = \eta\Phi^{3/4}$$

it is reduced to a system with a single parameter, which, naturally, is more convenient for analysis:

$$\frac{1}{2}\frac{d\varphi^2}{d\omega} = -C\varphi^2 + \frac{n-1}{n}\,(1-\varphi^4)^2, \tag{15}$$

$$\frac{d}{d\omega}\left(\frac{\varphi\psi}{n+1}\right) = -C\left(\frac{\varphi\psi}{n+1}\right) + \frac{n\varphi}{\psi},\quad C = \frac{\alpha\,(p)\,\Phi^{-3/2}}{R^2}, \tag{16}$$

$$\frac{n\varphi}{\psi} = 1 - \varphi^4. \tag{17}$$

We first consider the case where the influence of reradiation can be disregarded. Ignoring φ^4 as compared to unity, from (16) and (17) we obtain

$$\frac{d}{d\omega}\left(\frac{n}{n+1}\varphi^2\right) = -C\,\frac{n}{n+1}\,\varphi^2 + 1.$$

The solution of this equation satisfying the boundary condition $\omega = 0$, $\varphi = 0$ has the form

$$\frac{n}{n+1}\varphi^2 = \frac{1-\exp(-C\omega)}{C} . \tag{18}$$

Eliminating φ from (15), we obtain the equation for determining n:

$$\frac{d}{d\omega}\left(\frac{n+1}{n}\frac{1-\exp(-C\omega)}{C}\right) = -2\frac{n+1}{n}[1-\exp(-C\omega)] + 2\frac{n-1}{n} .$$

This equation is integrated and, on condition that $n \neq 0$ if $\omega \to 0$, gives

$$\frac{1}{n} = \frac{\exp(-C\omega)}{(\exp(C\omega)-1)^3}\left[\frac{1}{2}(\exp(C\omega)-1)^2 + (\exp(C\omega)-1) + C\omega\right]. \tag{19}$$

For $\omega \to 0$ we have $n \to 3$, which conforms with [1]. Using (18) we find

$$\varphi^2 = \frac{1}{C}\left[1-\exp(-C\omega) + \frac{1}{2}\exp(-2C\omega) - \frac{1-(1+C\omega)\exp(-C\omega)}{[\exp(C\omega)-1]^2}\exp(-C\omega)\right]. \tag{20}$$

The stationary distribution of the surface temperature is realized when $\omega \to \infty$:

$$\varphi_{\mathrm{st}} = \frac{1}{\sqrt{C}} \quad \text{or} \quad \theta_{\mathrm{st}} = \frac{R}{\sqrt{\alpha}}\exp(-p^2) .$$

The exact solution for the stationary state is given in [6]:

$$\theta_{\mathrm{st}} = \frac{\sqrt{\pi}}{2} R I_0\left(\frac{p^2}{2}\right)\exp(-p^2/2) . \tag{21}$$

Since the solution of the problem is approximate, for α we take the expression that gives the exact stationary temperature distribution over the surface of the plate:

$$\alpha = \frac{4}{\pi}\frac{\exp(-p^2)}{I_0^2(p^2/2)} .$$

At small times, the approximate model overstates the temperature at the center of the spot by 2.3% and, thereafter, predicts a more rapid establishment of its stationary level than the exact model. The limiting (stationary) distribution of the surface temperature in the presence of reradiation will be found from (15) on condition that $d\varphi/d\omega \to 0$ (in this case, $n \to \infty$):

$$\frac{\varphi_{\mathrm{st\,i}}^2}{(1-\varphi_{\mathrm{st\,i}}^4)^2} = \frac{1}{C} . \tag{22}$$

The algebraic equation of the fourth degree (22) has a unique real positive root:

$$\varphi_{\mathrm{st\,i}} = \sqrt{\sqrt{\frac{C}{8y}} - \frac{y}{2}} - \sqrt{\frac{y}{2}} ,$$

where

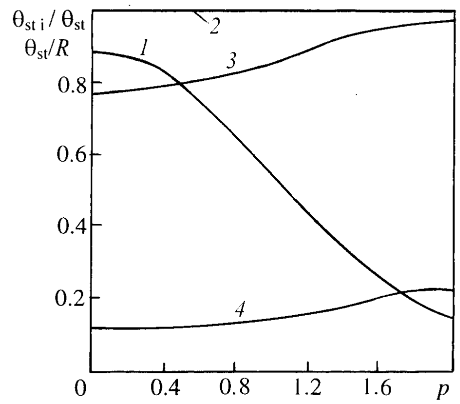

Fig. 1. Graphs of the stationary temperature distributions: 1) θ_{st}/R; 2–4) $\theta_{st\ i}/\theta_{st}$ [2) R = 0.1; 3) 1; 4) 10].

$$y = u + v\,;\quad u = \left(\sqrt{\left(\frac{C}{16}\right)^2 - \frac{1}{27}} + \frac{C}{16}\right)^{1/3};\quad v = -\left(\sqrt{\left(\frac{C}{16}\right)^2 - \frac{1}{27}} - \frac{C}{16}\right)^{1/3}.$$

Figure 1 shows the distribution of the limiting temperature over the radius $\theta_{st}(p)/R$ (curve 1) in the case where there is no radiation from the surface. It is seen from (21) that this temperature profile depends on only p. The distributions of the ratio of the stationary temperature in the case where there is radiation from the surface ($\theta_{st\ i}$) and in the case where it is absent (θ_{st}) (curves 2–4) are also presented in this figure. At small R, i.e., at small heat fluxes, the temperature of the radiating surface is practically equal to that in the case where radiation is absent (curve 2 corresponds to $\theta_{st\ i}/\theta_{st} \approx 1$). At $R \sim 1$, owing to the significant role of reradiation, it is much lower than the temperature of the nonradiating surface in the central part of the heated spot and approaches it at the periphery. At large R, the difference between them increases significantly (here $\varphi_{st\ i} \sim 1$).

Fusion of the Surface under Nonuniform Heating. Fusion of a material begins when the melting temperature T_m is attained at the center of the focal spot. The minimum value of the heat flux $q_{i0\ min}$ at which a point lying on a given radius is heated to T_m will be determined not only by the radiation-equilibrium temperature but also by the spreading of heat from the center of the spot in the radial direction. Let us find the quantity $q_{i0\ min}(p)$. Taking into account that in this case

$$\varphi_{st} = \frac{T_m}{T_*} \exp\left(\frac{p^2}{4}\right),$$

and substituting it into (22), we obtain the following expression for determining $q_{i0\ min}(p)$:

$$q_{i0\ min} = \left(\varepsilon\sigma T_m^4 + \frac{\lambda T_m}{r_*}\sqrt{\alpha}\right)\exp\,(p^2)\,. \qquad (23)$$

It is seen that the heat flux required to heat the surface to the melting temperature consists of two parts: the first part goes to compensate for the radiation from the surface heated to the melting temperature, and the second part goes to compensate for the heat spreading in the radial direction. As an example, in the calculations we used a ceramic specimen with the following thermophysical properties: ρ = 1500 kg/m^3, λ = 0.6 W/(m·K), ε = 0.8, c = 1.200 J/(kg·K), L_m = 1.5·10^5 J/kg, and T_m = 1900 K.

Figure 2 shows the dependence $q_{i0\ min}(p)$ for heating a ceramic plate to T_m at r_* = 1 cm. The minimum heat flux for the beginning of melting at the center $q_{i0\ min}$ is equal to 0.72 MW/m^2. The graph presented also allows one to find the limiting dimension of the melt hole p_{max}, which could be observed under infinitely long heating of the sur-

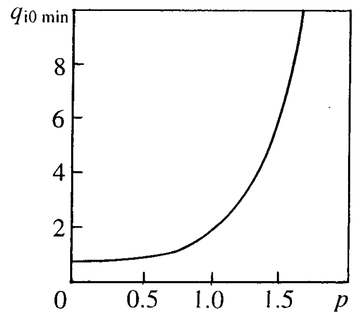

Fig. 2. Heat flux required for heating the points on the plate. $q_{i0\,min}$,

face by a given heat flux. For example, at q_{i0} = 2 MW/m^2, the maximum diameter of the melt hole would be somewhat larger than 2 cm.

Let us consider a situation where a melt film already exists. At each point on the boundary between the melt and the solid body, the condition of energy conservation is fulfilled:

$$q_{iw} = q_w + \rho L_m \frac{d\delta_m}{dt} . \tag{24}$$

We assume that the thickness of the melt film δ_m is small and, in this case, the temperature in the melt is distributed linearly along the coordinate z, i.e., the heat flux entering the melt reaches the boundary with the solid body without changes:

$$q_{iw} = q_{i0}\Phi (r) - \varepsilon\sigma T_0^4 = \lambda \frac{T_0 - T_m}{\delta_m} . \tag{25}$$

Equations (24) and (25) allow one to determine the growth of the melt film with time if the heat flux q_w is known.

We approximate the temperature profile by the function

$$T = T_m (1 - \zeta)^n , \quad \text{where} \quad \zeta = \frac{z - \delta_m}{\delta - \delta_m} , \tag{26}$$

and integrate Eq. (7) between the limits $\delta_m \le z \le \delta$. As a result, we will have

$$\frac{d}{dt} \frac{\delta - \delta_m}{n + 1} + \frac{a^2 \alpha (r)}{r_*^2} \frac{\delta - \delta_m}{n + 1} + V_m = \frac{a^2 n}{\delta - \delta_m} .$$

To obtain an equation for determining n, we substitute (26) into (7) and set $z = \delta_m$:

$$V_m = \frac{d\delta_m}{dt} = - a^2 \frac{\alpha (r) (\delta - \delta_m)}{n r_*^2} + a^2 \frac{n - 1}{\delta - \delta_m} .$$

Therefore,

$$n = \frac{1}{2}\left(1 + \frac{V_m (\delta - \delta_m)}{a^2}\right) + \sqrt{\frac{1}{4}\left(1 + \frac{V_m (\delta - \delta_m)}{a^2}\right)^2 + \frac{\alpha (r) (\delta - \delta_m)^2}{r_*^2}} .$$

It is well to bear in mind that passage from heating to melting is characterized by an abrupt change of boundary conditions on the surface of the solid body — from conditions of the second kind to conditions of the first kind. In this connection, in the present problem, one of the form parameters, namely the index n, changes spasmodically.

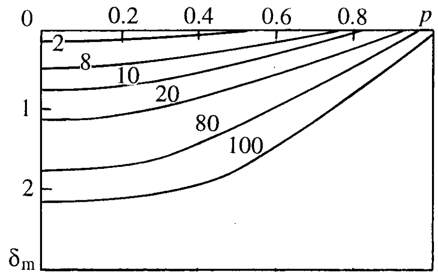

Fig. 3. Evolution of the melt hole on a ceramic plate with time for q_{i0} = $2 \cdot 10^6$ W/m^2 (figures on the curves mean the time in seconds passed from the beginning of heating). δ_m, mm.

The heat flux going into the solid body is equal to

$$q_w = \lambda \frac{nT_m}{\delta - \delta_m} .$$

Let us introduce the additional dimensionless variables

$$\eta_m = \frac{\delta_m}{l_*} ; \quad \theta_m = \frac{T_m}{T_*} ; \quad B_m = \frac{L_m}{cT_*}$$

As a result, we obtain the following system of equations to describe the process of melting:

$$\frac{\theta - \theta_m}{\eta_m} = \Phi (p) - \theta^4 , \tag{27}$$

$$B_m \frac{d\eta_m}{d\tau} = \Phi (p) - \theta^4 - \frac{n\theta_m}{\Delta\eta} , \tag{28}$$

$$\frac{d}{d\tau} \frac{\Delta\eta}{n + 1} = \frac{\alpha (p)}{R^2} \frac{\Delta\eta}{n (n + 1)} + \frac{1}{\eta} , \tag{29}$$

$$n = \frac{1}{2} \left(1 + \Delta\eta \frac{d\eta_m}{d\tau} \right) + \sqrt{\frac{1}{4} \left(1 + \Delta\eta \frac{d\eta_m}{d\tau} \right)^2 + \alpha (p) \Delta\eta^2} , \quad \Delta\eta = \eta - \eta_m . \tag{30}$$

Thus, the calculation of the process of fusion of a material consists of two stages. At the first stage, system (12)–(14) with initial conditions τ = 0, θ = 0, η = 0, and n = 3 is solved until the temperature of the surface along the given radius becomes equal to the melting temperature ($\tau = \tau_1$). At the second stage, we solve system (27)–(30) with the initial conditions, i.e., the values of θ and η at the instant of time τ_1, η_m = 0.

Figure 3 shows the evolution of the melt hole over a period of heating of 100 sec. During the initial period of time, the diameter of the hole rapidly increases and reaches the practically limiting value even by the 20th second. From this point on it only becomes deeper.

CONCLUSIONS

1. It is shown that the integral methods of calculation of the nonlinear problems of heat conduction can be used for approximate calculation of the process of heat treatment of plane products in solar furnaces with the Gaussian distribution of the radiant heat flux over the focal-spot area.

2. Simple formulas for estimating the limiting diameter of the melt hole and the minimum heat flux necessary for melting the surface of the product have been obtained.

3. The procedure for calculating the dynamics of heating and fusion of plane products in the focal spot of a solar furnace is described, and an example of calculation of the heat treatment of a ceramic plate is presented.

NOTATION

α, function introduced in Eq. (5); a^2, thermal-diffusivity coefficient, m^2/sec; B_m, dimensionless parameter of melting heat; c, heat capacity, J/(kg·K); η, dimensionless thickness of the heated layer of the solid material in the presence of a melt film; I_0, Bessel function of zero order; r and z, radial and axial coordinates; t, time, sec; T, temperature, K; T_m, melting temperature, K; λ, thermal-conductivity coefficient, W/(m·K); ρ, density, kg/m^3; L_m, melting heat, J/kg; θ and τ, dimensionless temperature and time; I_* and t_*, characteristic length and time, m and sec; T_*, radiation-equilibrium temperature, K; ε, radiating capacity; σ, Boltzmann constant, W/(m^2·K); δ, characteristic thickness of the heated layer, m; δ_m, thickness of the melt film, m; p, dimensionless radial coordinate; R, dimensionless parameter of the Gaussian distribution of the heat flux; r_*, parameter of the Gaussian distribution of the heat flux, m; q, heat flux, W/m^2; q_{i0}, radiation flux, W/m^2; q_{iw}, heat flux at the boundary between the melt film and the solid wall, W/m^2; V_m, rate of growth of the melt film, mm/sec; n, index of approximation of the temperature with respect to the coordinate; ω, φ, and ψ, dimensionless time, temperature and thickness of the heated layer in Eqs. (15)–(17). Subscripts: i, radiation; m, melting; 0, surface of the product or the melt film; w, boundary between the melt and the solid wall; z, in the direction of the axis; st, stationary.

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